E-gravity theory as a Yang-Mills theory

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Yang-Mills theories are a very fruitful concept in quantum field theory. Fundamental interactions and its unifications can be described with Yang-Mills theory. However, gravity is still not modeled in the framework of Yang-Mills theory. It is modeled in terms of the Einstein-Hilbert action in the case of semiclassical field theory, but ordinary quantization of the spacetime field fails due to UV divergences. A possible approach to quantum gravity called E-gravity theory avoids UV-divergences. Primary, this theory is based on a spacetime discretization and the assignment of a curvature measure to discretized spacetime. This paper shows that this approach is also a special case of Yang-Mills theory.

INTRODUCTION

The most plausible approach for making a model for fundamental elementary particle interaction is the Yang-Mills theory. In this theoretical framework, a local invariance of a physical action under a gauge group action is assumed. This invariance imposes the existence of a gauge connection that is also called “gauge boson field”. Electromagnetic, electroweak and strong interaction can be described very well in terms of Yang-Mills theory. Electroweak interaction is generated by imposing a $\mathfrak{su}(2)$ gauge group invariance and strong interaction can be obtained if the action functional is invariant under $\mathfrak{su}(3)$ group transformations. The Standard model of particle physics is based on the gauge group $\mathfrak{u}(1)\times\mathfrak{su}(2)\times\mathfrak{su}(3)$. May be $A_i$ a gauge connection in spacetime direction $\Gamma_i$ with generator index $a$, the structure constants of the gauge group, $g$ the coupling constant and $F_{ij} = \partial_i A_j - \partial_j A_i + g f^{abc} A_a A_b$ the associated field strength. Then the Yang-Mills Lagrangian density $L$ has the form:

$$L = \frac{1}{2} F_{\mu\nu}^a F^{\mu\nu}_a + L_{\text{ ghosts}}$$

The additional Lagrangian density $L_{\text{ ghosts}}$ is arising from the measure factor of the quantum path integral and contains the Faddeev-Popov ghost fields $c^\dagger_a$ that are Grassmannian variables. It holds:

$$L_{\text{ ghosts}} = c^\dagger_a \left( \delta^{ab} \partial^\mu A_b + g f^{abc} \partial^\mu A_b \right) c^a$$

However, gravity, also another fundamental force, cannot be described in terms of Yang-Mills theory. The only well-known consistent way for describing gravity is General Relativity, which is a classical theory of gravity. An open question is whether gravity can be formulated as a Yang-Mills theory. Several research papers show that there is existing a relationship between gravity and Yang-Mills theory (Bern, Z. et al. 1998) (Hsu, J.P. 2006).

There are existing theories that quantize gravity, e.g. Loop Quantum Gravity. The derivation of Loop...
Quantum Gravity is based on a reformulation of General Relativity by introducing $SU(2)$ gauge fields (Ashtekar, A. 1985). However, Loop Quantum Gravity does not coincide with Yang-Mills theory, e.g. due to the use of constraint terms and the quantization in terms of Wilson loops. Another recent approach to quantum gravity is E-gravity theory (Linker, P. 2016). This theory provides a discretization of spacetime with simplices. Simplices can have two or more simplex corners that are identified (in mathematical terms: there exists a nontrivial equaliser). Because of this additional property, a curvature measure can be assigned to each simplex. General Relativity is rewritten in this discrete form and a path integral over all possible spacetime geometries is performed.

In this research paper, an alternative derivation of E-gravity theory is shown. At first, spacetime is discretized by finite elements which are the simplices. Finite element methods are typically used for engineering problems, but are also applied to quantum field theories (Sopuerta, Carlos F. et al. 2005). Instead of using assumptions about the discretization of the Einstein-Hilbert action, a Yang-Mills theory with the diffeomorphism operator as the gauge group is worked out. Also, the field strength tensor, which is different from the smooth fiber bundle curvature form, is derived. With proper projection of the Yang-Mills theory to physical states, E-gravity theory is recovered.

THE DERIVATION OF E-GRAVITY THEORY BY YANG-MILLS THEORY

The theory of gravity is assumed to be a gauge theory with respect to the Poincaré group. If General Relativity is transformed locally by the Poincaré group, the action remains the same. Generators of the Poincaré group are local spacetime translations and local Lorentz transformations. Since the translation generators $t_\gamma = -i\alpha, \gamma^\lambda$ corresponding to a 4-vector $\gamma$ are operators and not finite-dimensional matrices, Yang-Mills theory in the form (1) cannot be applied. The goal is to find a matrix-like gauge group for gravity. The first step to find a proper gauge group is to use a finite element discretization of spacetime. In general, a physical field $\phi(x)$ (can be scalar, spinor, vector or tensor field) on a spacetime point $x$ is discretized as follows:

$$\phi(x) = \sum_{k=1}^{\mathcal{N}} \phi_k \sigma_k(x),$$

Here, $\phi_k$ are the field values at simplex corner (or node) $x_k$ and $\sigma_k(x)$ the corresponding form functions that have the value 1 at simplex point $x_k$ and the value 0 elsewhere. These form functions generate a discretized spacetime without self-intersection of simplices. Additionally, if two simplex nodes are identified (a key feature of E-gravity theory), the sum in equation (3) runs from $\mathcal{N}$ to $\mathcal{N} - \mathcal{M}$ with the number of identified nodes $\mathcal{M}$. Denotes $\mathcal{A}$ the set of all possible form functions that discretize the spacetime and $\mathcal{N}$ the set of all nodes in the spacetime, one obtains the path integral:

$$Z = \int \prod_{k \in \mathcal{A}} d[\phi_k] \prod_{l \in \mathcal{N}} d[\phi_l] \delta(\phi_1, \phi_2, \cdots, \phi_{\mathcal{N}})$$

Clearly, the set $\mathcal{A}$ is quite different from the simplex node equaliser set. However, the path integral (4) can be reduced to E-gravity theory by an appropriate gauge group. May the gauge operator $\chi$ acting on a function $\mathcal{F}(x)$ with the entire spacetime region $\Omega$ be defined as follows:

$$\chi(\mathcal{F}) = \int_{\Omega} d^4 y \mathcal{F}(x, y) R(y)$$

Because the operator (5) modifies the physical fields globally, the field $\chi$ is a global gauge
transformation. The operator (5) can be turned into a local gauge transformation by replacing
\[ \chi \rightarrow \chi(x, y) = \chi(x, y) \]

For a plausible model of gravity, the invariance of the action under the modified Poincaré group \( \mathfrak{P}(1, 3) \)
with group elements \( \chi \) acting on physical states in 4-dimensional spacetime is imposed. This group is
called “modified”, because all elements of the group acting like the convolution operator (5); these
groups are “infinite-dimensional matrices”. Moreover it holds \( \chi^\dagger \chi = 1 \), i.e. \( \chi \in \mathfrak{P}(1, 3) \) is an unitary
transformation. Due to linearity of \( \chi \) and the invariance of the action \( S[\phi, R \in \mathfrak{K}] \) under group
transformations in \( \mathfrak{P}(1, 3) \), the choice of form functions is arbitrary. If the deformation of simplices by \( \chi \)
is diffeomorphic, one can set without loss of generality \( \chi(x) = \chi(2) \), i.e. local and global gauge
transformations are the same. This can be easily seen if a differential operator \( \gamma \) with respect to \( \chi \) is
performed: By chain rule,
\[ \nabla \chi(x) = \nabla \chi(2) \] and equation (6) is only true if and only if \( \nabla \chi(x) |_{x, \gamma = 0} = 0 \). Hence, the gauge connection vanishes
in case of any diffeomorphic deformations.

If two or more simplex nodes are identified, one performs a non-diffeomorphic deformation. The left
hand side of equation (6) does not exist in general. By switching the gauge transformation from a
local to a global transformation, one can generalize the gauge group \( \chi \in \mathfrak{P}(1, 3) \) to a non-diffeomorphic
gauge group. This implies the existence of a non-smooth (!) gauge connection \( \zeta \). Clearly, a gauge
connection which is not differentiable, makes not sense in a physical field theory. If \( \zeta(x) = 0 \) on a
certain spacetime point \( x \), then \( \zeta(x) \) is singular, since \( \zeta(x) \) is necessary only for non-diffeomorphic
deformations. As a consequence, physical states can only be observed in regions with \( \zeta(x) \neq 0 \) for all
spacetime directions \( x \). Field strengths \( F_{\mu \nu} \) are not vanishing necessarily, because derivatives of \( \zeta \)
are not zero in general even if \( \zeta(x) \neq 0 \).

The path integral (4) can be reduced with the assumptions made above. Introducing a parameter
\( \zeta \in (0, 1) \) where \( \zeta \) denotes the index of the topological simplex structure (i.e. it denotes which nodes are
identified). This parameter runs over all different states of a simplex with same topology and therefore it
is called the “diffeomorphic deformation parameter”; in diffeomorphic deformation this parameter
changes. For non-diffeomorphic deformations, the gauge connection \( \zeta \) is changing. Therefore:
\[ J \frac{d[\xi]}{d[\zeta]} = \sum_{r \in T} \int_{\mathfrak{K}(1)} d[\zeta] \, J[\zeta] \cdot [\zeta = \zeta_{r}] \]

The set \( T \) is the set of all possible simplex topologies, i.e. the set of all identified simplex nodes.
Physical action does not change under diffeomorphic deformations (including Lorentz transformations)
and hence, the integral over \( \zeta \) for each \( r \in T \) is constant. The integration over all gauge connections
makes only sense for \( \zeta \rightarrow 0 \). All contributions with \( \zeta(x) = 0 \) can be neglected; only physical states are
taken into account, while the number of unphysical states is vanishing small. Hence, the Faddeev-
Popov factor in the path integral factorizes in the path integral and can be omitted. One is left with the
following path integral:
The path integral (8) is the path integral of E-gravity theory if the field strength term \( \frac{1}{2} tr(F \wedge F) \) corresponds to the Einstein-Hilbert action in vacuum. It holds for a 1-form gauge connection \( \omega = \frac{d\varphi}{2} \) and a 2-form field strength tensor \( F \) for \( \varphi = \varphi_0 \). Using theorem of Stokes over a 2-dimensional area \( \mathcal{A} \) one obtains
\[
\int_{\mathcal{A}} F = \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}'} \varphi.
\]
If \( \mathcal{A} \) is sufficiently small, i.e. \( |\mathcal{A}| \ll 1 \) (\( |\mathcal{A}| \) denotes the measure of a set), one can define the field strength corresponding to the area \( \mathcal{A} \):
\[
F_{\mathcal{A}} = \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}'} \varphi.
\]
From (9) it follows that if \( \mathcal{A} = \emptyset \) and the path integration is not performed around a singularity, it holds
\[
v_{\mathcal{A}} = 0.
\]
This is the case if there are no identified simplex nodes in a simplex element. The statement \( \delta \cdot \mathcal{A} = 0 \) with coboundary map \( \delta \), E-semigroup elements \( s_1, \ldots, s_l \) and indicator function \( t \) (that is 1 if there are no identified simplex nodes and 0 otherwise) is proven. If there are 2 or more identified simplex nodes (these induce singularities), one can choose closed paths in arbitrary spacetime directions such that all 3 or more singularities are contained in the path. Since \( \mathcal{A} \) is small, one has \( \mathcal{A} \approx \text{constant} \) everywhere on \( s_0 \) and in this case \( v_{\mathcal{A}} = 0 \).

It has to be shown that if exactly one simplex node \( s_k \) is identified with \( s_i \) and \( s_j \), it holds \( R = 0 \) if \( R \) is even and \( 1 \) is odd or \( R \) is odd and \( 1 \) is even, \( R = -k \) if \( 1 \) and \( 1 \) are odd; the value \( R \) is a specific gravitational constant. At first, the following decomposition of the gauge field action (which is known from electromagnetic theory) can be performed:
\[
S_{\text{grav}} = -\frac{1}{4} tr(F \wedge F) + \frac{1}{2} tr(F \wedge F) + \frac{1}{2} tr(F \wedge F) + \frac{1}{2} tr(F \wedge F).
\]

The indices \( i, j \) are running from 1 to 3 (all space directions). The term \( v_{ij} \) is the “gravito-electric” field strength and the term \( v_{ij} \) is the “gravito-magnetic” field strength. By Lorentz transformations, the time direction can be replaced by a direction pointing anywhere in spacetime, where all other indices denote all other directions perpendicular to the direction with index 0; action functional does not change after any Lorentz transformation.

Simplex elements have to be oriented properly while discretizing spacetime. Non-diffeomorphic deformations have singular values of the connection across the 0-direction, if the simplex node generators \( s_k \) or \( s_j \) are identified with another simplex node (by convention of the coboundary map, all nodes with odd index are negatively oriented), and have singular values of the connection perpendicular to this 0-direction, if all other simplex node generators \( s_k, s_k \) and \( s_k \) are identified with another simplex node (by convention of the coboundary map, all nodes with even index are positively oriented). Every singularity has the same magnitude of the value \( \frac{x}{y} \) given by \( \delta \). If \( K \) is even and \( J \) is odd or \( K \) is even and \( J \) is odd, the path integration passing through 0-direction encounters one singularity and the path integration passing through directions only perpendicular to 0-direction encounters also one singularity; hence \( \int_{\mathcal{A}} F_{\mathcal{A}} = \int_{\mathcal{A}} F_{\mathcal{A}} = K \) and \( s_{ijn} = 0 \).
If $\mu$ and $\lambda$ are even, the path integration in arbitrary 0-direction encounters two singularities of $\mathcal{A}$ with equal magnitude, while path integration in directions perpendicular to the 0-direction have no singularities; hence $\sqrt{\mathcal{L}} = 2\mathcal{K}_0$, $\mathcal{L}_0 = 0$ and $s_{\mathcal{L}_0} = c\mathcal{K}_0$ with a constant $c$. Finally, if $\mu$ and $\lambda$ are odd, the path integration in directions perpendicular to 0-direction encounters two singularities of $\mathcal{A}$ with equal magnitude, while path integration in directions in 0-direction have no singularities; hence: $\sqrt{\mathcal{L}} = 2\mathcal{K}_0$, $\mathcal{L}_0 = 0$ and $s_{\mathcal{L}_0} = -c\mathcal{K}_0$ with a constant $c$. The values of $s_{\mathcal{L}_0}$ are also infinite values, but with renormalization of the Yang-Mills coupling constant $g$ this infinity can be removed.

In four spacetime dimensions, the Ricci scalar defined by $\mathcal{R} = 2\mathcal{L}(\mathcal{S}_{\mathcal{L}_0})$ (averaged over a volume) can be also recovered by a Yang-Mills theory. Some idealizations like the equal values of a topological singularity are assumed, which might lead to small inaccuracies if the field strengths are computed by more advanced concepts (not treated in this research paper). Inhomogeneity in spacetime, which is also a feature of E-gravity theory, is caused exactly by the field $\mathcal{A}$ which is nonvanishing between the simplex elements, where physical states can occur. This gauge connection separates simplex elements which are different. E-gravity theory in terms of a Yang-Mills theory would conserve energy, momentum and spin-angular momentum, but if the 1-form field $\mathcal{A}$ is “cut out”, energy and momentum has to be created and annihilated to include effects performed by the gauge connection. Gravitational interactions are able to materialize multiple particles. From the Yang-Mills theory, gravity couples on the currents (here, the fermionic case with fermion fields $\mathcal{A}$ and coupling constant $g$)

$$\mathcal{L} = g^2 \mathcal{A}^* \gamma^i \gamma^j \mathcal{A}_i$$

where $\gamma^i$ is the generator of diffeomorphic deformations. It is a convolution operator and from (11) it holds that gravity is induced by correlations of particles in the entire spacetime. A simple example is the correlation of an apple that correlates with the earth; the apple falls towards earth because gravity is triggered by the apple-earth correlation. In systems where quantum entanglement can occur, physical states can correlate; therefore gravity is also produced in entangled systems.

CONCLUSIONS

An interesting correspondence between Yang-Mills theory and General Relativity is shown in this research paper. Yang-Mills theory with another version of the Poincaré group, called $\text{SU}(2,\mathbb{C}) \times \text{SU}(1,1)$ with special assumptions about simplex topology turns out to coincide with the Einstein-Hilbert action that uses a generalized curvature measure based on chain complexes of simplices. Formulating fundamental interactions in terms of Yang-Mills theory makes it easier to find unified descriptions of fundamental forces. Indeed, the gauge group of Standard model of particle physics combined with E-gravity theory can be regarded as the gauge group $\text{SU}(2,\mathbb{C}) \times \text{SU}(1,1) \times \text{SU}(1,1)$. Unfortunately, a plausible Grand Unified Theory is still not found. However, E-gravity theory also applies to effects, where not all other fundamental forces are unified.


